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Photoionization of a One-Electron *N*-Dimensional Atom

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We consider the photoionization of a one-electron atom in *N*-dimensional space and derive the total cross-section. It is shown that $\sigma(\omega)$ depends strongly on the dimension *N*. Thus we emphasize the role of the topological structure of a system on the physical properties of the system.

1. INTRODUCTION

Nowadays it is well recognized that the concept of dimensions plays an important role in theoretical and mathematical physics. Zeng *et al.* (1994, 1997) discussed the transformation between a hydrogen atom and a harmonic oscillator of arbitrary dimensions. Neves and Wotzasek (2000) considered the quantization of a free particle on an *N*-dimensional sphere, using the Stuckelberg field-shifting formalism. Grosche and Steiner (1987) presented a general framework for treating path integrals on curved manifolds in *N*-dimensional polar coordinates. Periwal (1995), motivated by studying the strong coupling expansion without perturbation theory, proposed a formula for continuing physical correlation functions to higher dimensions. Craco and Laad (2000) studied electron energy-loss spectroscopy of strongly correlated systems in infinite dimensions. Wo'dkiewic (1991) investigated the theory of zero-range potentials in an arbitrary number of dimensions. Shrock and Wu (2000) presented a formulation for the enumeration of spanning trees in *N* dimensions. Al-Jaber (1998a) studied the fine structure of the *N*-dimensional hydrogen atom. Romeo (1995) offered some insight into the dimensional dependence of the Wentzel-Kramers Brillouin approximations in connection with hyperspherical quantum billiards. Al-Jaber (1999) considered Fermi gas in *N*-dimensional space. Sophocleous (2000) described a class of discrete symmetries for *N*-dimensional nonlinear wave equations that form, in some cases, cyclic groups of finite order.

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In this contribution we consider photoionization of a one-electron atom in *N*-dimensions. So the purpose of this paper is to derive the transition rate of absorption of radiation by a one-electron *N*-dimensional atom. We also calculate the cross-section of the photoionization process.

2. THE TRANSITION RATE

Let us consider the ejection of an electron from the ground state of a one-electron *N*-dimensional atom, by the absorption of radiation of angular frequency ω, or in other words by the absorption of a photon of energy $\hbar\omega$. The electric dipole approximation will be made and this is expected to be accurate if the wavelength of the radiation is greater than atomic dimensions. The wave function and energy of the initial (ground) state of the *N*-dimensional atom are (Al-Jaber, 1998b)

$$\Psi_a(\vec{r}) = \frac{A_0}{[\Omega(N)]^{1/2}}\, e^{-2zr/a_0(N-1)} \tag{1}$$

$$E_a = \frac{-Z^2e^2}{4\pi\varepsilon_0(2a_0)\left(N-\frac{1}{2}\right)^2} \tag{2}$$

where Ze is the nuclear charge, a_0 is Bohr radius, and we assumed the infinite nuclear mass approximation. The normalization constant A_0 and the volume of the *N*-dimensional unit sphere $\Omega(N)$ are given by

$$A_0 = \left[\left(\frac{4Z}{a_0(N-1)}\right)^N \frac{1}{(N-1)\{(N-2)!\}^3}\right]^{1/2} \tag{3}$$

$$\Omega(N) = 2\pi^{N/2}/\Gamma(N/2). \tag{4}$$

The final atomic state b represents a free electron of momentum $\hbar\vec{K}_b$ moving in the Coulomb field of the nucleus. In the nonrelativistic regime (i.e. $\hbar\omega$ or $E_b \ll mc^2$) we have

$$E_b = \hbar^2K_b^2/2m. \tag{5}$$

Assuming that E_b is sufficiently large with respect to the binding energy $|E_a|$ of the electron, so that the Coulomb interaction with the nucleus can be neglected, the wave function of the final continuous state is approximated by a plane wave

$$\Psi_b(\vec{K}_b, \vec{r}) \cong \frac{1}{(2\pi)^{N/2}}\, e^{i\vec{K}_b\cdot\vec{r}}. \tag{6}$$

In the dipole approximation the electric field vector, $\vec{E}$, is taken to be independent of position and the interaction with the atom can be written as

$$H'(t) = -\vec{E}\cdot\vec{D}. \tag{7}$$

Where $\vec{D}$ is the electric dipole moment of the system and $\vec{E}$ is the electric field that satisfies Maxwell's equations and thus has the form

$$\vec{E} = E_0(\omega)\hat{\varepsilon}\ \sin(\vec{k}\cdot\vec{r} - \omega t + \delta), \tag{8}$$

where $\hat{\varepsilon}$ is the polarization vector. It follows from the dipole approximation that the field can be taken to be uniform over the whole atom, and equal to its value at the nucleus, which we take to be at the origin. Thus

$$H'(t) = \hat{\varepsilon}\cdot\vec{D}E_0(\omega)\ \sin(\omega t - \delta) \tag{9}$$

Equations (7) and (9) enable us to write $H'(t)$ as

$$H'(t) = A\, e^{i(\omega t-\delta)} + A^+\, \vec{e}^{\,i(\omega t-\delta)}, \tag{10}$$

where A is a time-independent operator given by

$$A = \frac{1}{2i}\hat{\varepsilon}\cdot\vec{D}E_0(\omega)\,\vec{e}^{\,i\delta}, \tag{11}$$

and A^+ is its adjoint operator.

The transition probability per unit time (transition rate) is (Bransden and Joachain, 1990)

$$W_{ba} = \frac{2\pi}{\hbar}|A^+_{ba}|^2\rho_b(E_f), \tag{12}$$

where $E_f = E_a + \hbar\omega$ and $\rho_b(E_f)$ is the density of final states. The first-order total transition rate is thus

$$W_{ba} = \frac{2\pi}{\hbar}\frac{E^2(\omega)}{4}\rho_b(E_f)\int \cos^2\theta\,|\vec{D}_{ba}|^2 d\Omega, \tag{13}$$

where $d\Omega$ is a solid angle in N-dimensional space, which is given by (Shimakura, 1992)

$$d\Omega = \prod_{j=1}^{N-2}(\sin\theta_j)^{N-1-j}\,d\theta_j\,d\varphi. \tag{14}$$

In our present case the final states are plane waves, so that if $\rho_b(E_f)\,d\Omega\,dE_f$ denotes the number of states whose wave vector $\vec{K}_f$ lies within the solid angle $d\Omega$ and the energy of which is in the interval $(E_f,\ E_f + dE_f)$, we have

$$\rho_b(E_f)\,d\Omega\,dE_f = d\vec{K}_f = K_f^{N-1}\,dK_f\,d\Omega, \tag{15}$$

and thus, using Eq. (5), we obtain

$$\rho_b(E_f) = \frac{m}{\hbar^2}K_f^{N-2}. \tag{16}$$

It is well known that the intensity of the electromagnetic field is (Griffiths, 1999)

$$I(\omega) = \frac{1}{2} c\varepsilon_0 E_0^2(\omega). \tag{17}$$

The substitution of Eqs. (16) and (17) into Eq. (13) yields

$$W_{ba} = \frac{\pi m k_f^{N-2}}{\hbar^3 c\varepsilon_0} I(\omega) \int \cos^2\theta |\vec{D}_{ba}|^2 \, d\Omega \tag{18}$$

3. CROSS-SECTION

The absorption cross-section σ_{ba} for the transition from a to b is defined by (Bransden and Joechain, 1990)

$$\sigma_{ba} = (\hbar\omega_{ba}) W_{ba}/I(\omega_{ba}), \tag{19}$$

and using Eq. (18) we see that

$$\sigma_{ba} = \frac{\pi m\omega}{\hbar^2 c\varepsilon_0} k_f^{N-2} \int \cos^2\theta |\vec{D}_{ba}|^2 \, d\Omega \tag{20}$$

It is helpful to use the velocity form of the dipole matrix element $\vec{D}_{ba}$ that is given by (Bransdena and Joechain, 1990)

$$\vec{D}_{ba} = \frac{ie}{m\omega} \vec{P}^*_{ab}, \tag{21}$$

which, upon the use of Eqs. (1) and (6), becomes

$$\vec{D}_{ba} = \frac{ie\hbar}{m\omega} \frac{A_0}{(2\pi)^{N/2}} \frac{\vec{k}_f}{(\Omega(N))^{1/2}} \int \vec{e}^{2z/a_0(N-1)} \, \vec{e}^{i\vec{k}_f\cdot\vec{r}} \, dr. \tag{22}$$

The integral on the right of Eq. (22) is the Fourier transform of $\vec{e}^{2z/a_0(N-1)}$ and thus

$$\int \vec{e}^{2z/a_0(N-1)} \, \vec{e}^{i\vec{k}_f\cdot\vec{r}} \, dr = \frac{8\pi(2Z/a_0(N-1))}{\left[\left(\frac{2Z}{a_0(N-1)}\right)^2 + k_f^2\right]^2}. \tag{23}$$

Substituting this result in Eq. (22), we get

$$\vec{D}_{ba} = \frac{ie\hbar}{m\omega} \frac{1}{(2\pi)^{N/2}} \frac{A_0}{(\Omega(N))^{1/2}} \vec{k}_f \frac{16\pi Z/a_0(N-1)}{\left[\left(\frac{2Z}{a_0(N-1)}\right)^2 + k_f^2\right]^2}, \tag{24}$$

and therefore the cross-section becomes

$$\sigma(\omega) = \frac{256\pi^3 e^2}{mc\varepsilon_0\omega} \frac{A_0^2}{(2\pi)^N} \frac{k_f^N}{\Omega(N)} \frac{Z^2/a_0^2(N-1)^2}{\left[\left(\frac{2Z}{a_0(N-1)}\right)^2 + k_f^2\right]^4} \int \cos^2\theta \, d\Omega. \tag{25}$$

It is assumed that the energy of the ejected electron is much greater than the binding energy $|E_a|$ of the electron. This implies that

$$\frac{\hbar^2 k_f^2}{2m} \cong \hbar\omega \Rightarrow k_f^2 \cong 2m\omega/\hbar. \tag{26}$$

Furthermore, the condition $\hbar\omega \gg |E_a|$ implies that, from Eq. (2), we can neglect $(\frac{Z}{a_0})^2$ in comparison with k_f^2. Therefore

$$\sigma(\omega) = \frac{2^{10}\alpha\hbar\pi^4}{m\omega(2\pi)^N}\frac{A_0^2}{\Omega(N)}k_f^{N-8}\frac{Z^2}{a_0^2(N-1)^2}\int \cos^2\theta\, d\Omega, \tag{27}$$

where $\alpha = e^2\hbar c/4\pi\varepsilon_0$ is the fine structure. Let us consider the direction of propagation of radiation as the z-axis and the polarization vector $\hat{\varepsilon}$ to be along the X_{N-1} *axis*. We notice from Eq. (24) that $\vec{D}_{ba}$ is in the direction of $\vec{k}_f$ which is defined by the polar angles $(\theta_1, \theta_2, \ldots, \theta_{N-2}, \varphi)$ and we see that the x_{N-1} component of $\vec{k}_f$ is given by (Grosche and Steiner, 1987)

$$x_{N-1} = r\left(\prod_{j=1}^{N-2}\sin\theta_j\right)\cos\varphi,$$

and thus

$$\cos^2\theta = \left(\prod_{j=1}^{N-2}\sin^2\theta_j\right)\cos\varphi. \tag{28}$$

The substitution of Eq. (28) into Eq. (27) yields

$$\sigma(\omega) = \frac{2^{10}\alpha\hbar\pi^4 A_0^2}{m\omega(2\pi)^N\Omega(N)}\, k_f^{N-8}\, \frac{Z^2}{a_0^2(N-1)^2}\int \prod_{j=1}^{N-2}\sin^2\theta_j\, \cos^2\varphi\, d\Omega. \tag{29}$$

For an unpolarized photon beam an average must be made over all directions of the polarization vector $\hat{\varepsilon}$, which implies an average over the $\cos^2\varphi$ factor and this gives a factor $1/2$. Thus, in that case

$$\sigma(\omega) = \frac{2^9\alpha\hbar\pi^4 A_0^2\, k_f^{N-8}}{m\omega(2\pi)^N\Omega(N)}\frac{Z^2}{a_0^2(N-1)^2}\int \prod_{j=1}^{N-2}(\sin\theta_j)^{N+1-j}\, d\theta_j\, d\varphi, \tag{30}$$

where we have used Eq. (14). Using Eqs. (3), (4), and (26) and noticing that the integration over the angle φ gives a factor of 2π, we can write Eq. (30) as

$$\sigma(\omega) = \frac{2^{(3N+10)/2}}{\pi^{(3N-10)/2}}\,\alpha\left(\frac{Z}{a_0(N-1)}\right)^{N+2}\left(\frac{\hbar}{m\omega}\right)^{(10-N)/2}$$
$$\times\frac{\Gamma(N/2)}{(N-1)\{(N-2)!\}^3}\int\prod_{j=1}^{N-2}(\sin\theta_j)^{N+1-j}\, d\theta_j \tag{31}$$

The integral in Eq. (31) can be carried out by using the formula (Boas, 1983)

$$\int_0^{\pi} (\sin\theta)^{2p-1}\, d\theta = B\left(p, \frac{1}{2}\right), \tag{32}$$

where B(p, q) is the beta function. Thus

$$\int_0^{\pi} (\sin\theta_j)^{N+1-j}\, d\theta_j = B\left(\frac{N-j+2}{2}, \frac{1}{2}\right). \tag{33}$$

Substituting Eq. (33) into Eq. (31), we obtain

$$\sigma(\omega) = \frac{2^{(3N+10)/2}}{\pi^{(3N-10)/2}}\alpha\left(\frac{Z}{a_0(N-1)}\right)^{N+2}\left(\frac{\hbar}{m\omega}\right)^{(10-N)/2} \times \frac{\Gamma(N/2)}{(N-1)((N-2)!)^3}\prod_{j=1}^{N-2} B\left(\frac{N-j+2}{2}, \frac{1}{2}\right). \tag{34}$$

It is clear that the total cross-section $\sigma(\omega)$ increases with Z as Z^{N+2}. Also, it is noticed that as ω increases the total cross-section, $\sigma(\omega)$ decreases for $N < 10$ and increases for $N > 10$, so that $\sigma(\omega)$ does not change for $N = 10$. It is instructive to consider the three-dimensional case ($N = 3$), in this case, Eq. (34) gives us

$$\sigma(\omega) = \frac{16\sqrt{2}}{3}\pi\alpha\left(\frac{Z}{a_0}\right)^5\left(\frac{\hbar}{m\omega}\right)^{-7/2}, \tag{35}$$

which is the well-known result (Bransden and Joechain, 1990).

4. CONCLUSION

In this paper we have shown that the dimension N of space plays an important role in determining the physical properties of a system. In particular, we have studied the photoionization of a one-electron atom in N dimensions. The total cross-section $\sigma(\omega)$ for absorption was derived and it was noticed that $\sigma(\omega)$ depends strongly on N, especially for the Z and ω dependence. Furthermore, our result for $\sigma(\omega)$ reduces to the expected result in three-dimensional space.

REFERENCES

Al-Jaber, S. (1998a). The fine structure of the N-dimensional hydrogen atom. *Nuovo Cimento B* **113**, 651.

Al-Jaber, S. (1998b). Hydrogen atom in N dimensions. *International Journal of Theoretical Physics* **37**, 1289.

Al-Jaber, S. (1999). Fermi gas in D-dimensional space. *International Journal of Theoretical Physics* **38**, 919.

Boas, L. M. (1983). *Mathematical Methods in the Physical Science*, Wiley, New York.

Bransden, B. and Joachain, C. (1990). *Introduction to Quantum Mechanics*, Longman Scientific & Technical, England.

Craco, L. and Laad, M. (2000). Electron energy-loss spectroscopy of strongly correlated systems in infinite dimensions. *Journal of Physics: Condensed Matter* **12**, 7647.

Griffiths, D. J. (1999). *Introduction to Eelctrodynamics*, Prentice-Hall, Englewood Cliffs, NJ.

Grosche, C. and Steiner, F. (1987). Path integrals on curved manifolds. *Zeitschrift fur Physik C* **36**, 699.

Neves, C. and Wotzasek, C. (2000). Stuckelberg field-shifting quantization of a free particle on D-dimensional sphere. *Journal of Physics A: Mathematical and General* **33**, 6447.

Periwal, V. (1995). Dimensional continuation without perturbation theory. *Modern Physics Letters A* **10**, 1195.

Romeo, A. (1995). Multidimensional extension of a Wentzel-Kramers-Brillouin improvement for spherical quantum billiard zeta functions. *Journal of Mathematical Physics* **36**, 4005.

Shimakura, N. (1992). *Partial Differential Operators of Elliptic Type*, American Mathematical Society Providence, Raode Island.

Shrock, R. and Wu, F. (2000). Spanning trees on graphs and Lattices in d dimensions. *Journal of Physics A: Mathematical and General* **33**, 3881.

Sophocleous, C. (2000). On cyclic symmetries of n-dimensional nonlinear wave equations. *Journal of Physics A: Mathematical and General* **33**, 8319.

Wo'dkiewic, K. (1991). Fermi psedopotential in arbitrary dimensions. *Physical Review A* **43**, 68.

Zeng, G., Su, L. K., and Li, M. (1994). Most general and simplest algebraic relationship between energy eigenstates of a hydrogen atom and a harmonic oscillator of arbitrary dimensions. *Physical Review A* **50**, 4373.

Zeng, G., Zeng, S., Zeng, A., and Jiang, F. (1997). Transformation between a hydrogen atom and a harmonic oscillator of arbitrary dimensions. *Journal of Physics A: Mathematical and General* **30**, 1775.